CLIFFORD ALGEBRA IN DIFFERENT SPACES AND ITS APPLICATIONS

Submitted by

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Abstract

The fundamentals of spaces are covered in functional analysis. The properties and Mathematical definitions of integral transforms and Clifford algebra are presented in chapter 1.

The concept of a Clifford Fourier transform with the new kernel was proposed in an earlier study with linear property, additivity property, and other properties. Some fundamental properties of Clifford transform and fractional Clifford transforms, such as convolution, Plancherel, and vector differential theorems, were established in chapter 2.

Clifford-wavelet transform in L^2-spaces is defined in space-time algebra $Cl_{(3,1)}$ of Minkowski space with orthonormal vector basis. The properties of the Clifford-wavelet transform are established. Plancherel's theorem and reproducing kernel are demonstrated. The inversion formula for the Clifford-wavelet transform is established. The study is supported with examples and applications from Mathematical Physics in chapter 3.

In chapter 4, two-sided Clifford-wavelet function (CWF) is defined in $Cl_{(n_1,n_2)}$ with an orthonormal vector basis. The properties like left linearity, right linearity, shifting, modulation dilations, and power factor are established. The inversion formula for CWF is also constructed using the Fourier transform. Parseval and Plancherel identities are also studied. The study is supported by certain examples related to Mathematical Physics.

Two-dimensional Clifford fractional Mellin transform (2D-CFrMT) is defined in the space-time algebra $Cl_{(3,1)}$ with an orthonormal vector basis. The properties of 2D-CFrMT are established. The inversion formula for 2D-CFrMT is also constructed. Parseval's type property and convolution theorem are studied. The graphical interpretations

with an example are represented in chapter 5. The study is supported by applications in Mathematical Physics.

The two-sided generalized Clifford fractional Mellin transform (2D-GCFrMT) is defined in the space-time algebra. The testing function space is established.

Clifford algebra is formulated using an orthonormal basis in $Cl_{(3,1)}$. Maxwell's equations are represented using fractional Clifford Mellin transform with multivectors in chapter 6. The basic structure of Clifford algebra in space-time is studied in chapter 7. The idempotent properties using the similitude group are studied. 2D-CFrMT is considered part of the applications of the Maxwell equation in electromagnetism in this chapter.

In this study, the Dirac equation and spinors are developed in Clifford algebra $Cl_{(3,1)}$ using the representation of the Gamma matrices 8. Gamma and Pauli matrices in $Cl_{(3,1)}$ are used to show applications in Clifford algebra using the Clifford-wavelet function.

An exact analytical representation of the Clifford Hankel transform, as a series of the Bessel functions multiplied by the Clifford wavelet coefficients of the input function, is obtained in 9. Numerical examples are given to illustrate the proposed algorithm.

FCMT is defined in $Cl_{(3,1)}$ and applied to various functions. The developed kernel is approximated, and numerical evaluations are obtained in 9. Results and conclusions of all the chapters are summarized in 10. Some recommendations for further research based on our contributions are stated in chapter 11.

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CHAPTER 1

INTRODUCTION

1.1 Preliminary Results

Basic definitions of topological spaces, sequence, multinorm, inner product, and continuous linear mapping are stated. The sections contain all the basic definitions of integral transforms, Clifford algebra, and various mathematical concepts that will be required throughout the thesis.

1.2 Objective of the work

The motive of the research work is to develop Clifford algebra $Cl_{(3,1)}$ and $Cl_{(n_1,n_2)}$ with an orthonormal vector basis. Study similitude group and spaces like L^1, L^2 and Minkowski in Clifford algebra $Cl_{(3,1)}$ and $Cl_{(n_1,n_2)}$. The study initially developed analysis of Clifford-wavelet transform in $Cl_{(3,1)}$ and further generalized to 2D-GCFrMT in $Cl_{(n_1,n_2)}$. The study also aims to construct CFrMT and its generalization in $Cl_{(3,1)}$. The inversion formula of the hence-developed transforms is constructed.

Further, the Clifford wavelet and the new two-dimensional Clifford fractional Mellin transform are applied to quantum mechanics and electromagnetism theory. The objective to demonstrate the applications using CWT, CWF, 2D-CFrMT, and 2D-GCFrMT in Mathematical Physics and Engineering are met.

1.3 Scope of the Thesis

Clifford algebra $Cl_{(3,0)}$-valued classical wavelet scalar functions can be extended to Clifford algebra multivector functions. Properties such as translation, dilation, rotation covariance, and reproducing kernel for multivector functions were demonstrated in $Cl_{(3,0)}$ [1], can be extended to multivector Clifford wavelet $Cl_{(3,1)}$ and the properties in space-time algebra and the insights w.r.t. orthonormal basis.

The algebraic identities in inner and outer products are presented in [2], which can be aimed to develop a Clifford-wavelet transform using a similitude group. In the study [3], the two-dimensional quaternion wavelet transform $Cl_{(2,0)}$ has been introduced in continuous quaternion wavelet transform, which can further be studied in $Cl_{(3,1)}$.

Brackx et al. [4] extended the Fourier transform to multivector-valued function and distributions in $Cl_{(n,0)}$ with compact support. Generalization of wavelet transforms [1] can be adopted in Clifford algebra $Cl_{(3,1)}$. The detailed properties of the developed transform can be established in 3.

CA wavelet in the forthcoming chapter establishes a general form of a new Fourier transform of Clifford wavelet function in $Cl_{(n_1,n_2)}$, analogous to the scalar formulation extended from chapter 3, where the admissibility constant is scalar. [5] introduced basic concepts of the multivector functions, vector differential, and vector derivative in geometric algebra in $Cl_{(3,0)}$ which can be extended to $Cl_{(n_1,n_2)}$.

In [6], a non-commutative generalization of the complex Fourier-Mellin transform to CA valued signal functions over the domain $\mathbb{R}^{n_1,n_2}$ taking values in $Cl_{(n_1,n_2)}, n_1+n_2=2$ are illustrated. Working with $Cl_{(n_1,n_2)}$ allows for a unified treatment of algebra with different space-like and time-like signatures to apply CA to different physical contexts and mathematical structures. Different combinations of n_1 and n_2 and $Cl_{(n_1,n_2)}$ support different geometric properties, and flexibility allows applications in areas where specific geometric interpretations are needed like Riemannian geometry and complex manifolds

in 4 Additionally square root of (-1) in $Cl_{(n_1,n_2)}$ allows us to construct Fourier of Clifford wavelet function, [7] provides an overview of real geometric square roots of (-1) in CA of real vector spaces $\mathbb{R}^{n_1,n_2}$, includes real and complex numbers, quaternions, Pauli and Dirac algebra, space-time, spinor algebra, Lie algebra and conformal geometric algebra.

The Clifford Fourier-Mellin transform of a real signal in $Cl_{(1,1)}$ automatically separates symmetry components concerning the reflection of the unit circle and the reversal in the sense of rotation. Beyond this, the total modulus remains invariant for scaling and shifting, forming the foundation for translation, rotation, and scale-invariant shape recognition in the next chapter. The four components of the CFrMT kernel differ by radial and angular phase shifts as mentioned in [6] motivated the next chapter, where several properties of CFrMT, including inversion, convolution, and Parseval, are stated. Further, E. Hitzer [7] has studied symmetry properties of the Clifford Fourier-Mellin transform derivatives, power scaling, Plancherel, and Parseval theorems. The study limits to a part of the approach in CFT and the applications addressed, whereas the next chapter formalizes applications involving PDE for CFrMT in $Cl_{(3,1)}$ in the chapter 5.

The fractional Clifford Mellin-based correlators can detect or control a specific range of movements. In the Mellin-based correlators, the motion is assumed to consist of rotation and scaling [8], which can now be used to compute space-time motivating for the forthcoming chapter. The common correlators provide information regarding the translation of a certain object or its scaling and rotation in $Cl_{(1,1)}$. The correlators are based on the concepts of fractional transformations like fractional Fourier transform [9], [10] and fractional Mellin transform that can detect movements in a specific range in $Cl_{(2,0)}$ chapter 6.

The fractional Mellin-based correlator can control the range of rotation and scaling. In contrast, the generalized fractional Mellin transform can also be used to control the range of translation. The two-dimensional CFrMT is generalized in the distributional

sense. Its analyticity and boundedness theorem is proved. The inversion formula theorem and uniqueness theorem are also proved and thus can be used in navigational systems as a co-relator to control moments in specific ranges. GCFrMT in $Cl_{(3,1)}$ is applied to solved BVP in the generalized sense, which was solved in [11] in the classical sense.

7 The study in [12] contributed to the Dirac equation and is fundamental to Quantum Mechanics in terms of CA $Cl_{(3,0)}$. It describes the behavior of spin $\frac{1}{2}$ particles like electrons in relativistic quantum field theory. Development of the Mathematical structure of spinors in CA and their transformations were studied earlier in [13]. The sense of motivation to introduce the Dirac equation using CWF was developed in 3. Further study explains spinors and chiralities in space-time correlating CA and CWF in $Cl_{(3,1)}$. CWF that have been established in 3 is applied to various transformations to see the behavior change which was lagging in analyzing in earlier work in $Cl_{(3,1)}$.

8 The study in CFrMT is developed in 5 and 6 in $Cl_{(3,1)}$, which motivated us to look at the finite aspects of Mellin transform transform. It also aims to approximate the kernel and is interested in seeing the exact and approximated value of FCMT.

CHAPTER 2

LITERATURE REVIEW

In 1763, Euler introduced integral transforms while studying second-order differential equations [11]. 1800 French mathematician and physicist Jean-Baptiste Joseph Fourier developed the Fourier transform [14]. The concept of wavelets started to appear in the literature only in the early 1980s. In 1984, Jean Morlet first applied wavelets to analyze seismic reflection signals. In [15], the continuous wavelet transform overcomes this by providing localized frequency resolution in the phase space. It was Alex Grossmann, a French theoretical physicist, who quickly recognized the importance of the Morlet wavelet transform, which is something similar to coherent states formalism in Quantum Mechanics, and developed an exact inversion formula for the wavelet transform.

Hjalmar Mellin is recognized for creating the Mellin transform in 1897 [8]. Frigyes Riesz first introduced Lp spaces in 1910. In [16] author has discussed about the Lp in details. [17] In 1927, Wolfgang Pauli published a quantum theory of an electron in an electromagnetic field. Paul Dirac published a relativistic quantum theory of electrons in an electromagnetic field a year later. The author compares the vector and geometric algebra formulations of Pauli's and Dirac's theories in the study.

1878 William Kingdome Clifford introduced Clifford algebra [18]. In [17], the author discussed the development and progress of Clifford algebra. In [12] and [19], the author has given the development and construction of Clifford algebra. Geometric calculus was developed [20] motivated by the fact that two vectors can be multiplied. [21] authors

have explored Clifford algebra to provide a natural alternative to Minkowski's formulation with new insights into the nature of time. Authors in this paper given applications of orthogonal transformations, spinors and matrices, applied geometric calculus, Physics, Geometric algebra, the representation of higher order curves and surfaces [22]. [23] have shown how continuous Clifford $Cl_{(3,0)}$ valued admissible wavelets were constructed using the similitude group SIM(3), a subgroup of the affine group of $\mathbb{R}^3$. In 2006, [24] constructed the Clifford algebra-valued admissible wavelets associated with more than 2-dimensional Euclidean groups with dilations. Admissibility conditions, inversion formula, and Plancherel's theorem were established in [1]. In [25], authors have considered Clifford-valued functions defined on $\mathbb{R}^n$ with representation in square-integrable group. In the study of two-dimensional quaternion wavelet transform [3], authors have introduced continuous quaternion wavelet transform (CQWT) and established the admissibility condition in terms of the right-sided quaternion Fourier transform, norm relation, and inversion formula. In 2011, [26] proposed a two-dimensional continuous quaternion wavelet transform using the orthogonality of harmonic exponential functions and an alternative proof for inner product relation. In [27], Clifford multivectors have been studied for a new description of space and time. Authors have shown Clifford geometric algebra utilizing multivectors to represent space-time, providing a compact Mathematical representation and insights into the nature of time. Author in [28] has studied the admissibility condition in terms of a $Cl_{(n,0)}$ Clifford-Fourier transform, dilation, translation, rotation covariance, reproducing kernel, and inversion formula of Clifford-wavelet transform.

The extensions of classical work of wavelet transform to Clifford algebra versions of wavelet transform are shown in [20]. Clifford (Geometric) algebra wavelet transform was introduced, and continuous $Cl_{(n,0)}$ valued admissible wavelets were constructed using the similitude group SIM(n).

Quaternion Fourier transform on quaternion fields, and generalizations were introduced

in [29]. In [30], the author has inspired the straightforward definition of a general geometric Fourier transform covering most versions in the literature. In [31], the author has discussed the development of wavelet transform and its properties. In [32], Clifford algebra offers a geometric interpretation for square roots of (-1) in the form of blades that square to (-1), which extends to a geometric interpretation of quaternions as represented as bivectors of a unit cube.

In [33], the author has explained the orthogonal planes split of quaternions based on the arbitrary choice of one or two linearly independent pure unit quaternions f, g. Also, the author has generalized the quaternionic Fourier transform applied to the quaternion field to determine by quaternion f, g, and established inverse transformation. In [3], the author has found that Physics and the Mathematical structure that arises from the geometry of the underlying space-time is that of Clifford algebra. In [6], the author described a non-commutative generalization of the complex Fourier-Mellin transform to Clifford algebra valued functions over the domain $\mathbb{R}^{(n_1,n_2)}$ taking values in $Cl(p,q),\ p+q=2$. In [34], the author has generalized the classical Fourier-Mellin transform and investigated the properties of the quaternionic Fourier-Mellin Transform. In [27], authors have illustrated how Clifford's Geometric algebra utilizes multivectors to represent space-time, providing an elegant Mathematical framework for studying relativistic phenomena.

In [35], the Mellin transform method is applied to fractional differential equations with a right-sided derivative and variable potential.

In 2009, [8] two-dimensional fractional Mellin transform was studied. In 2012, authors developed the theory of Clifford algebra in space-time [27]. [6] author described in 2012 a non-commutative generalization of the complex Fourier-Mellin transform to Clifford algebra valued functions. The author investigated the properties of the QFMT [34]. Author [17] introduced geometric calculus and its related properties in 2017. Authors in [36] have established a two-dimensional quaternionic Mellin transform in a distri-

butional sense in 2021. Authors [36] have developed two-dimensional Clifford fractional Mellin transform (2D-CFrMT) [37] in the space-time algebra $Cl_{(3,1)}$. Generalized Mellin transform were represented in [38]. In [39], the application of differential properties to solve the solution of the linear partial differential equation in Clifford analysis is shown.

In [40], the authors have introduced space-time algebra, which emphasizes the practicality and power as a tool to study electromagnetism. The space-time concept studied in [27] describes the idea of Clifford multivectors. The authors have shown Clifford Geometric algebra utilizing multivectors to represent space-time, providing a compact Mathematical representation and insights into the nature of time. In [23], authors have shown continuous Clifford $Cl_{3,0}$ in similitude group SIM(3), which is a subgroup of the affine group of $\mathbb{R}^3$. In [2] and [18], the authors discussed the development and progress of geometric algebra. In [40], the authors have given the general construction of alternative algebra with a unit and three anticommuting elements. Authors have considered a version of the fractional Clifford-Fourier transform (FrCFT) [39] and studied several properties and applications to partial differential equations in Clifford analysis. For embedding one-dimensional signals in $\mathbb{R}^2$, considered the special case of $n = 2$, and in [41] and [41] this was also applied to the processing of two-dimensional vector fields. Recent applications with $n = 2, 3$ to geographic information systems and climate data can be found in [42].

In [27], authors have discussed Clifford algebra using multivectors. Clifford wavelet in $Cl_{(3,0)}$ using similitude group SIM(3) is constructed, and properties of Clifford wavelet transform using multivector function are shown in [1]. In [24], authors have considered Clifford algebra-valued admissible wavelets and proved properties of the wavelet transform using admissible wavelets. Clifford-valued function defined on $\mathbb{R}$ is considered in [25] for representing continuous wavelet transform.

As the zero-order Hankel transform naturally arises in a variety of applications of tech-

nological interest, including optics [43] and [44] acoustics [45], electromagnetics [46] and image processing [47] and the method [48] faced difficulties in its evaluation

CHAPTER 3

ANALYSIS OF CLIFFORD-WAVELET TRANSFORM IN $Cl_{(3,1)}$

The admissibility condition in terms of $Cl_{(n,0)}$ Clifford Fourier transform, dilation, translation, rotation covariance, reproducing kernel, and inversion formula of Clifford-wavelet transform in [28] are illustrated. The study is explored in $Cl_{(3,1)}$ In this chapter in CA, an orthonormal basis vector is used. A multiplication table using basis blades is obtained. Linear combinations of multivectors using anti-automorphism to find a geometric product of two vectors and reverse relations are also demonstrated.

Multi-valued functions are used to develop CFT in space-time. The similitude group using rotors is calculated with all six angles, which have a unique Euler angle representation that was not studied earlier. The relationship between spin and similitude groups is also established. CWF uses a family of wavelets with rotation, dilation, and translation parameters stated, and the results are evaluated using two-sided FT.

The study also led to show that the Clifford wavelet function is independent of the dilation parameter while being normalized and to ensure further the existence of an inversion formula in space-time $Cl_{(3,1)}$. The admissibility condition is established. CWT is defined using Clifford-mother-wavelet and hence rotation covariance, translation, dilation, left and right linearity. The rate of change of CWT is also estimated. Plancherel's and Parseval's theorem are established in Clifford's sense. Inner product relation using admissible wavelet is developed in L^2 space.

Examples of CWF are solved, and a graph of real, vector, bi-vector, tri-vector, and

quadra-vector components of each example is plotted. The transform developed in this chapter studied in $Cl_{(3,1)}$ has been applied in CWT to the fermionic field using the equation of motion and Klein–Gordon equation in $Cl_{(3,1)}$.

CHAPTER 4

FOURIER OF TWO-SIDED CLIFFORD-WAVELET FUNCTION IN $Cl_{(n_1,n_2)}$

In [49], two-dimensional quaternion fractional FT was developed, and the properties of quaternion-valued functions were established for $Cl_{(2,0)}$. The present chapter extends to $Cl_{(n_1,n_2)}$, demonstrating the development of a two-sided Clifford-wavelet function and how a function can be decomposed in the Clifford algebra sense. The similitude group is extended to order n with the particular orthogonal group. The multivector is split into commuting and anti-commuting w.r.t to the square root of (-1) to apply in Fourier of CWF using the scalar product. Every multivector has a unique decomposition, which is shown in this chapter.

Properties like left and right linearity, shifting, dilation, modulation, and Power factor of Fourier of CWF are studied in $Cl_{(n_1,n_2)}$. Parseval, Plancherel, and Inversion of Fourier of CWF in L^1 space are constructed. Haar and Mexican hat wavelets are defined in Clifford's sense; hence, the Fourier transform was applied to find the solution. The developed transform solves the IVP in Clifford Schwartz space, which helped visualize the component-wise changes.

CHAPTER 5

CLIFFORD FRACTIONAL MELLIN TRANSFORM IN $Cl_{(3,1)}$ WITH APPLICATIONS AND GRAPHICAL INTERPRETATION

A version of the fractional Clifford-Fourier transform (FrCFT) was developed in [39] $Cl_{(2,0)}$, and the properties studied were further applied to partial differential equations. The study motivated to define CFrMT in space-time algebra using an orthonormal basis. The relation between CFrFT and CFrMT is observed in this chapter. The properties like left and right linearity, shifting, and scaling are studied with multivector functions in space-time $Cl_{(3,1)}$. The study further led to obtaining an inversion of CFrMT by applying the relation between CFrFT and CFrMT. Parseval and convolution theorem for 2D-CFrMT are also established. The induction method is applied to find the higher-order derivatives of the multi-valued function, and the w.r.t. time and spatial vector are observed. Graphs of the respective kernel are plotted by analyzing the real, vector, bi-vector, tri-vector, and quadra-vector components.

An example of CFrMT is illustrated, and hence, the developed transform is applied to a partial differential equation using the Heaviside step function and potential in an infinite wedge. Current in the electric circuit leads to the Euler differential equation, which is solved by applying CFrMT.

CHAPTER 6

2D-GENERALIZED FRACTIONAL CLIFFORD MELLIN TRANSFORMATION IN $Cl_{(3,1)}$

The earlier chapter developed 2D-CFrMT in $Cl_{(3,1)}$, which motivated the generalized two-sided CFrMT. The idea of generalized Mellin transform is given in [11], which was extended to CA of testing-function space using norms and seminorms in Fréchet space in $Cl_{(3,1)}$.

The kernel of transform is developed using dual-space of all smooth functions. The analyticity theorem is established in the generalized sense, which leads to the construction of the inversion formula of 2D-GFrMT in space-time. The anticommuting nature of CA in $Cl_{(3,1)}$ of the developed transform, including power factor property, is proved. Continuous and convergent functions in dual space are used to develop a Convolution-type theorem. Derivative properties using linear maps and adjoint operators are established. 2D-GFrMT is applied in Clifford's sense to solve BVP.

CHAPTER 7

CLIFFORD-WAVELET FUNCTION AND MELLIN TRANSFORM IN CLIFFORD ALGEBRA $Cl(3,1)$ WITH INTERPRETATION IN QUANTUM MECHANICS

In chapter 3 CA is defined in $Cl_{(3,1)}$. In this chapter, the commuting and anticommuting properties of CA are proved in detail by taking four abstract bases. Idempotent properties of the Clifford-wavelet function are shown, which was developed 3. The interpretation of Clifford-wavelet in Quantum Mechanics is established in $Cl_{(3,1)}$. The transform established in 5 is now applied in two-state quantum and Pauli matrices defined in CA $Cl_{(3,1)}$.

CHAPTER 8

DIRAC EQUATION AND THEIR SPINORS USING CLIFFORD ALGEBRA $Cl_{(3,1)}$ APPROACH

CWF defined in 3 is further analyzed component-wise using a family of Clifford wavelet in $Cl_{(3,1)}$, and MATLAB code is given to plot components like a scalar, vector, bi-vector, tri-vector, and quadra-vector of the wavelet function. Dirac equations are developed using CWF defined 3, and hence, the solution to the Dirac equation is given by finding eigenvalues and vectors. As shown in this chapter, gamma matrices mentioned in 7 also satisfy anticommutation relations. Further Gamma matrices are plotted to show all components in $Cl_{(3,1)}$ spinors using CWF are also presented. Lorentz transformation using rotors is given with the plot in CA $Cl_{(3,1)}$. Majorana spinors relation is given with charge conjugation in the CA sense. Supersymmetry and its transformation are expressed in this chapter. Gauge using spinor of CWF is expressed with coupling constant. Chiral symmetry and the related transformation are represented using the Dirac spinor's left and right components.

CHAPTER 9

NUMERICAL COMPUTATIONS IN CLIFFORD ALGEBRA

With a focus on its application to special functions like the Meijer G-function and generalized hypergeometric functions, the image's caption explains the Mellin transform method and how it can be used to evaluate definite integrals. It emphasizes the rarity and complexity of analytical evaluations of some circumstances, such as the finite quaternion Mellin transform, which drives the use of numerical techniques. A numerical method for the finite quaternion Mellin transform is introduced in the chapter, seemingly using "Clifford hat functions." It highlights the method's effectiveness and stability, noting that error analysis and graphical depiction are also included. It also discusses applications in general relativity, electrodynamics, Euler-Cauchy differential equations, and boundary value problems.

CHAPTER 10

SUMMARY AND CONCLUSION

Definitions and basic ideas of Mathematical concepts used throughout the study, including notations and terminology, are given in chapter 1. Chapter 2 is about conducting a systematic literature review, which begins with research reports. This chapter examines the work of well-known mathematicians on the subject and summarizes research papers. In chapter 3, multiplication of basis vectors is explored and developed CWT in space-time algebra $Cl_{(3,1)}$. The properties of CWT are studied. Plancherel's and inversion's formulas have been established. The study is supported with examples and applications in the Fermionic field and Klein–Gordon equation in $Cl_{(3,1)}$.

Two-sided Clifford wavelet function (CWF) is defined in $Cl_{(n_1,n_2)}$ with orthonormal vector basis in chapter 4. The properties like left and right linearity, shifting, modulation dilations, power factor, inversion formula, and Parseval and Plancherel identities are established. Fourier of CWF is applied to solve PDE with IVP in the CA sense. Chapter 5, 2D-CFrMT is developed in the present study. Properties based on 2D-CFrMT are demonstrated. The inversion formula and Parseval's type properties are established. The examples related to the hence-developed results are discussed. The graphical representation and application in the Heaviside step function, the potential of an infinite wedge, and electrical circuits applying 2D-CFrMT are presented in the work. 2D-GCFrMT in chapter 6 is defined in the space-time algebra $Cl_{(3,1)}$. The properties like left linearity, right linearity, scaling, shifting, derivative properties, inversion formula, Convolution-

type, and Parseval's type property of 2D-GCFrMT are established. BVP is solved in the CA sense by applying 2D-GCFrMT in $Cl_{(3,1)}$. Clifford-wavelet and idempotent properties are studied in 7. Clifford-wavelet is analyzed in the Quantum Mechanics field. Maxwell's equations are represented using Clifford fractional Mellin transform with multivectors spectral weight function. Applications in electromagnetism are demonstrated using 2D-CFrMT. Dirac equation and in CA $Cl_{(3,1)}$ using the representation of the gamma matrices are studied in 8. Graphs related to CWF and programs based on primary vectors are shown using MATLAB. The CWF is applied to various transformations to observe the outcome using CA $Cl_{(3,1)}$. Further in 9 FCMT in $Cl_{(3,1)}$ is developed and approximated. The error is calculated after approximation with an exact value.

CHAPTER 11

FUTURE WORK

The work related to Clifford-wavelet transform in L^2-spaces in space-time algebra $Cl_{(3,1)}$ may be extended to $Cl_{(n_1,n_2)}$.

Researchers can use the framework to handle multidimensional data more effectively while dealing with deep learning models.

Investigating how Clifford algebra can enhance convolutional neural networks or graphical neural networks.

The Clifford Mellin transform has a broad scope for future research, particularly in areas that deal with multidimensional data, complex geometric transformations, and real-time processing requirements.

The study can further be developed using different functions by applying the Clifford Mellin transform. Graphical interpretation and tabular data with such developed results can be interesting for researchers to analyze and relate to the recent contributions in Mathematical Physics.

Researchers in the future can use the hence-developed approach in the present study by the authors in Clifford algebra $Cl_{(3,1)}$ to establish the Dirac equation with their spinors in Quantum Mechanics, in particular, electromagnetism theory and its applications.

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Journals

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3. Shabnam Jahan Ansari and V. R. Lakshmi Gorty, *Two-sided Generalized Clifford Mellin transform*, Communicated (2024).

4. Shabnam Jahan Ansari and V. R. Lakshmi Gorty, *Dirac equation and their spinors using Clifford algebra* $Cl_{(3,1)}$ *approach*, Communicated (2024).

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1. Shabnam Jahan Ansari and V. R. Lakshmi Gorty, *Two-sided Clifford wavelet function in Cl(p,q)*, 7^{th} International Arab Conference on Mathematics and Computations (IACMC2022), proceedings in Springer journal, between 11-13 May, 2022 at Zarqa University, Jordan, Print ISBN 978-981-99-0446-4, Online ISBN 978-981-99-0447-1. https://doi.org/10.1007/978-981-99-0447-1_25 https://iacmc.zu.edu.jo/images/files/program3.pdf

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